Dynamical paths and universality in continuous variables noisy channels

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We introduce an overcomplete parameter space for two-mode symmetric Gaussian states suitable to address the decoherence effects of both Markovian and non-Markovian Gaussian maps. We observe universality of the dynamical paths, which do depend only on the initial state and on the effective temperature of the environment, with the non Markovianity that manifests itself in the velocity of running over a given path. Universality is also seen in the value of discord at the separability threshold and may be exploited to build constants of motions valid for both Markovian and non-Markovian maps. We also found that the geometrical constraints provided by the structure of the parameter space imply the existence of excluded regions, i.e. sets of states which cannot be linked by any Gaussian dynamical map.

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As soon as entanglement has been recognized as a resource for quantum technology, it has been realized that decoherence is the main obstacle to overcome. Decoherence appears whenever a system interacts with its environment, so that its dynamics is no longer unitary, but rather described by a noisy quantum map, irreversibly driving the system towards relaxation and loss of quantum coherence [1, 2]. The main effect of the interaction with environment is to set up a time scale τ_M over which the dynamics of the system is effectively described by a coarse grained Markovian process towards equilibrium. Conversely, for times shorter than τ_M, the dynamics is more involved and the correlations with and within the environment play a major role [2–6]. In this regime the decoherence may be less detrimental, or even almost unitary, and this is why a great attention has been devoted to the study of the corresponding *non-Markovian* maps. Indeed, there are evidences that non-Markovian open quantum systems [7–10] can be useful for quantum technologies [11, 12]. As a consequence, much attention is currently devoted to the analysis of system-environment coupling in order to characterize, control, and possibly reduce decoherence in the most effective way [13, 14], e.g. by taking advantage of the back-flow of information from the environment.

As a matter of fact, non-Markovian models are more involved and only few cases can be solved analytically [2, 15, 16]. However, these cases are also of great general validity, and they can display all of the properties of interest. This is especially true for continuous variable systems [17], where a set of quantum oscillators excited in a Gaussian state, and interacting with their thermal environment, provides an excellent model for a large class of physical systems in order to study non-Markovianity and the decoherence of quantum correlations. Motivated by the above considerations, we address in details the dynamics of quantum correlations between two quantum oscillators prepared in a symmetric Gaussian state and interacting with local thermal environments. At first, we notice that the set of Gaussian states [18] do not constitute a manifold, nor it is convex, and thus geometrical approaches to their dynamics are not considered particularly appealing. At variance with this belief, we address the study of decoherence by representing dynamical paths in a suitable, overcomplete, parameter space, involving entanglement, Gaussian discord and the overall purity of the state. These variables proved to offer a suitable framework to compare non-Markovian maps and their Markovian counterparts, and to show which properties do, and *notably do not*, distinguish Markovian and non-Markovian processes. In particular, we observe universality of the dynamical paths, which do not depend on the specific features of the environment spectrum and are determined only by the initial state and the effective temperature of the environment. The non-Markovianity of the system only changes the velocity of running over a given path. This behavior allows ones to map non-Markovian processes onto Markovian ones and it may reduce the number of parameters needed to study a dynamical process, e.g. it may be exploited to build constants of motions valid for both Markovian and non-Markovian maps. Universality is also observed in the value of discord at the separability threshold, which moreover is a function of the sole initial conditions in the limit of high temperature. Finally, we find that the geometrical constraints provided by the structure of the parameter space implies the existence of excluded regions, i.e. sets of states which cannot be linked by any Gaussian dynamical map.

Dynamical evolution—We address the dynamical decoherence of two oscillators of frequency ω_0, each coupled to its own bosonic environment made of modes at frequencies ω_k. The baths are separated and of equal structure, such that the evolution preserves any initial symmetry between the two oscillators. The system-bath interaction Hamiltonian is given by $H_I = \alpha\Sigma_k j_k(\hat{X}_1\hat{q}_{1,k} + \hat{X}_2\hat{q}_{2,k})$ where α is the cou-

pling, the complex $\{j_k\}_{k=1,2..}$ modulate the dispersion over the bath's modes, and $\hat{X}_i = (\hat{a}_i + \hat{a}_i^\dagger)/2$, $i = 1, 2$ and $\hat{q}_{k,i} = (\hat{b}_{k,i} + \hat{b}_{k,i}^\dagger)/2$ denote the canonical operators of the systems' and baths' modes respectively. In the weak coupling limit $\alpha \ll \omega_0$, and discarding the counter rotating terms, we get a non-Markovian master equation for the dynamical evolution of the density operator ϱ describing the quantum state of the two oscillators. Assuming that the initial state is a two-mode Gaussian state with zero amplitude the master equation may be transformed into an evolution equation for the two-mode covariance matrix [19, 20] (CM) σ, $\sigma_{jk} = \frac{1}{2}\mathrm{Tr}[\varrho(R_j R_k + R_k R_j)]$, $\mathbf{R} = \{X_1, Y_1, X_2, Y_2\}$, whose solution may be written as

$$\sigma_t = e^{-\Gamma(t)}\sigma_0 + \Delta_\Gamma(t)\,\frac{\mathbb{I}_4}{2} \tag{1}$$

where σ_0 is the initial CM,

$$\Gamma(t) = \int_0^t ds\,\gamma(s), \quad \Delta_\Gamma(t) = e^{-\Gamma(t)}\int_0^t ds\,e^{\Gamma(s)}\Delta(s)$$

and where, upon defining the spectrum of environment as $j(\omega) = \Sigma_k |j_k|^2 \delta(\omega - \omega_k)$, the diffusion and dissipation functions are given by [21]:

$$\begin{aligned}\Delta(t) &= \alpha^2 \int_0^t ds \int_0^\infty d\omega\, j(\omega)\coth(\omega\beta/2)\cos(\omega s)\cos(\omega_0 s)\\ \gamma(t) &= \alpha^2 \int_0^t ds \int_0^\infty d\omega\, j(\omega)\sin(\omega s)\sin(\omega_0 s)\end{aligned} \tag{2}$$

where $\beta = 1/kT$. At high temperatures the damping is negligible and the first coefficient is dominant, while at lower temperatures they have the same order of magnitude.

The non-Markovian features are embodied in the time dependence of the coefficients $\Delta_\Gamma(t)$ and $\Gamma(t)$, describing diffusion and damping respectively, which for times $t \lesssim \tau_{\scriptscriptstyle M}$ are strongly influenced by the whole spectrum of the environment [22, 23]. On the other hand, for times $t \gg \tau_{\scriptscriptstyle M}$ the coefficients achieve their Markovian limiting values. In particular we have $\lim_{t\to+\infty}\gamma(t) = \alpha^2|j(\omega_0)|^2 \equiv \gamma_{\scriptscriptstyle M}$, such that $\Gamma(t) = \gamma_{\scriptscriptstyle M} t$, $\Delta_\Gamma(t) = (1 - e^{-\gamma_{\scriptscriptstyle M} t})(2n_{\scriptscriptstyle T} + 1)$, and the solution (1) rewrites as $\sigma(t) = e^{-\gamma_{\scriptscriptstyle M} t}\sigma_0 + (1 - e^{-\gamma_{\scriptscriptstyle M} t})\sigma_{\scriptscriptstyle T}$, where $\sigma_{\scriptscriptstyle T} = (n_{\scriptscriptstyle T} + \frac{1}{2})\mathbb{I}$ is the CM of the stationary state, i.e. a state in thermal equilibrium with the environment and a population of $n_{\scriptscriptstyle T} = (e^{\beta\omega} - 1)^{-1}$ thermal photons.

Symmetric Gaussian states—Symmetric Gaussian states are those with a CM that can be recasted (via local operations) in a form depending on two real parameters $\sigma(a, c) = a\,\mathbb{I}_4 + c\,\sigma_1 \otimes \sigma_3$, the σ_j's being Pauli matrices. This correspond to preparing the two oscillators in a *squeezed thermal state*, with density operator of the form $\varrho(r,)n_{\scriptscriptstyle T} = S(r)\nu \otimes \nu S^\dagger(r)$, where ν is a single mode thermal state with $\nu_{\scriptscriptstyle T}$ photons and $S(r) = e^{r(a_1 a_2 + a_1^\dagger a_2^\dagger)}$ is the two-mode squeezing operator. The link with the parameter of the CM is given by $a = (\nu_{\scriptscriptstyle T} + \frac{1}{2})\cosh(2r)$ and $c = (\nu_{\scriptscriptstyle T} + \frac{1}{2})\sinh(2r)$. The diagonal elements $a = \frac{1}{2} + \bar{n}$ depends on the (equal) population (mean photon number) of the two subsystems $\bar{n} = \sinh^2 r(2\nu_{\scriptscriptstyle T} + 1) + \nu_{\scriptscriptstyle T}$, while the c coefficients describe the correlations among them. Uncertainty relations impose a constraint which reads [24] $|a - c| \geqslant \frac{1}{2}$.

As a matter of fact, the representation in terms of the coefficients a and c does not fully illustrate the correlation properties of a state. In particular, it does not allow to analyze the relations between different kinds of quantum correlations, as entanglement or discord, in a dynamical context. To this aim we introduce a different (overcomplete) parametrization involving the overall purity of the state $\mu = \mathrm{Tr}[\varrho(r, n_{\scriptscriptstyle T})^2]$, its entanglement expressed in terms of the minimum symplectic eigenvalue $\lambda = a - c$ (the state is separable iff $\lambda \geq \frac{1}{2}$) and the Gaussian quantum discord, which for symmetric Gaussian states, may be written as [25, 26]

$$D(a, c) = h(a) - 2h\left(\sqrt{a^2 - c^2}\right) + h\left(a - \frac{2c^2}{1 + 2a}\right) \tag{3}$$

where $h(x) = (x + \frac{1}{2})\log(x + \frac{1}{2}) - (x - \frac{1}{2})\log(x - \frac{1}{2})$. Of course, the resulting parameter space is overcomplete and the third parameter is a function of the other two [27]. In the following, we will describe the dynamical paths in the three-dimensional space (μ, λ, D). These will lie over the surface individuated by the constraint $D = D(\mu, \lambda)$, and in the region satisfying the uncertainty relations, which rewrite as $\mu < \frac{1}{4}\lambda^{-2}$.

Markovian dynamics—The Markovian master equation depends on the environment (effective) temperature and on the damping $\gamma_{\scriptscriptstyle M}$, nonetheless the Markovian dynamical paths depend exclusively on the (effective) temperature of the environment. The damping only affects the speed of running over a dynamical path, but not its shape, and the rate $c(t)/c_0 = e^{-\gamma_{\scriptscriptstyle M} t}$ determines in a unique way the rate $a(t)/a_0$. In the left panel of Fig. 1 we report Markovian paths for different temperatures, assuming the two oscillators initially prepared in a two-mode squeezed vacuum (TWB) $\varrho(r_0, 0)$, i.e a pure maximally entangled state. As it is apparent from the plot, two limiting paths emerge at low and high temperatures. The transition from one regime to the other occurs continuously by raising the temperature, and we see that the high temperatures limit is already achieved for temperatures leading to $n_{\scriptscriptstyle T}/\sinh^2(r_0) \gtrsim 3$. Two other phenomena are revealed by this representation: (i) the value of the discord at the separability threshold ($\lambda = \frac{1}{2}$) depends only on the initial squeezing r_0 and approaches a universal curve in the high temperature limit; (ii) for a given initial state $\varrho(r_0, 0)$ there are STS that cannot be reached during any Markovian decoherence process, despite the fact that they have reduced entanglement and purity compared to the initial state.

Non Markovian dynamics— As mentioned in the introduction, non-Markovian dynamics may display remarkable differences from their Markovian counterpart during the initial transient when $t \lesssim \tau_{\scriptscriptstyle M}$. Entanglement oscillations may occur, and the separability threshold may be delayed or accelerated depending on the spectrum of the environment. A question thus arises on whether these differences also affect significantly the dynamical path in the space of parameters. As

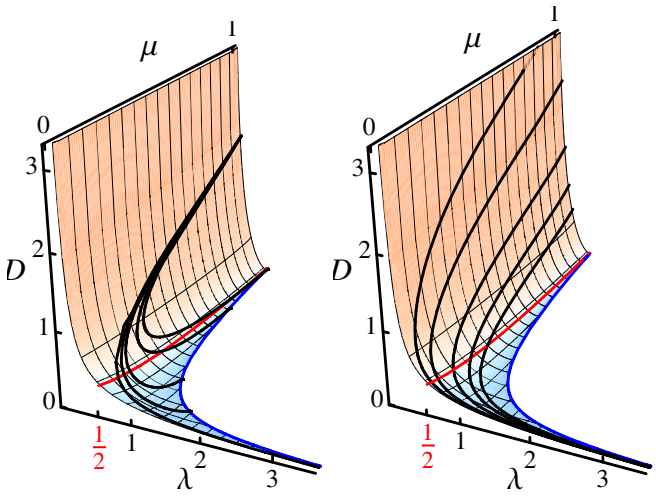

FIG. 1: Left: Markovian thermalization paths at different temperatures. The initial state is a STS with $r_0 = 1.2$, the different paths correspond to $n_T = 0, 0.1, 0.5, 1, 10$. Solid blue line correspond to thermal states with zero discord, solid red line denotes the separability threshold. The high-temperature limit is already achieved for $n_T \gtrsim 3$. Right: dynamical paths in the high-temperatures limit for different values of initial entanglement. The curves correspond to $r_0 = 0.5, 0.7, 1, 1.5, 2$.

we will see, the answer is negative, and universality occurs. The results about the dynamics that we are going to discuss are independent of the particular choice of environment spectrum. However, in order to show some numerical solutions, we employ few examples corresponding to white noise and to both Ohmic and super-Ohmic spectral densities with cut-off ω_c [28]. Let us start by analyzing the high temperature regime, where over a timescale $\tau \sim \tau_M$ we can neglect the damping $\Gamma(t)$ (it becomes relevant over times $\tau \sim \gamma_M^{-1} \gg \tau_M$, which is definitely in the Markovian regime). Short time non-Markovian dynamics is thus due to the behavior of the heating function $\Delta_\Gamma(t)$ and, in turn, is very sensitive to the details of the environment spectrum $j(\omega)$. In this limit non-Markovian effects can be seen during the whole decoherence process, with entanglement oscillation across the separability threshold [19]. The dynamics is driven by the approximate dynamical equation

$$\sigma_t \simeq \sigma_0 + \int_0^t ds\, \Delta(s)\, \frac{\mathbb{I}_4}{2} \tag{4}$$

corresponding to $a(t) = a_0 + \frac{1}{2}\int_0^t d\tau\, \Delta(\tau)$ and $c(t) = c_0$. The last condition imposes a constraint to the dynamical paths, which is the same independently on whether the dynamic of $a(t)$ is Markovian or displays oscillations, as long as $a(t) \geq a_0\ \forall t$ and $a(t) \to a_T$. In other words, the paths are the same of the Markovian case, and the possible oscillations of $a(t)$ only influences the speed of running over the dynamical path. In the right panel of Fig. 1 we show the dynamical paths for different values of the initial squeezing r_0.

The condition $c(t) = c_0$ also implies that the Gaussian discord may be written as $D(a, c) \simeq D(\lambda + c_0, c_0)$, i.e. it depends on the temperature and on the initial squeezing [29]. At the separability threshold we have $D_{sep} = D(\frac{1}{2}[1 + \sinh(2r_0)], r_0) \equiv f(r_0)$, i.e. the discord at separability is a universal function of the initial squeezing. In Fig.2 we show the Gaussian discord at separability as a function the initial squeezing. The solid black line correspond to the high temperature approximation $f(r_0)$, whereas the colored dots are obtained using the full non-Markovian solutions, obtained taking into account the damping and different environment spectra. We first point out that there is excellent agreement among the two solutions already for temperature about $n_T \lesssim 1$ and independently on the environment spectrum. We also notice that D_{sep} saturates to a limiting value $d_* = \lim_{r_0\to\infty} D(\frac{1}{2}[1 + \sinh(2r_0)], r_0) = -1 + 2\log 2 \simeq 0.3863$ when the initial squeezing increases. The initial squeezing needed for achieve the saturation regime increases with temperature. As it may be seen from the plot, for high temperatures, i.e. for $n_T \gtrsim 1$, it is about $r_0 \simeq 2$.

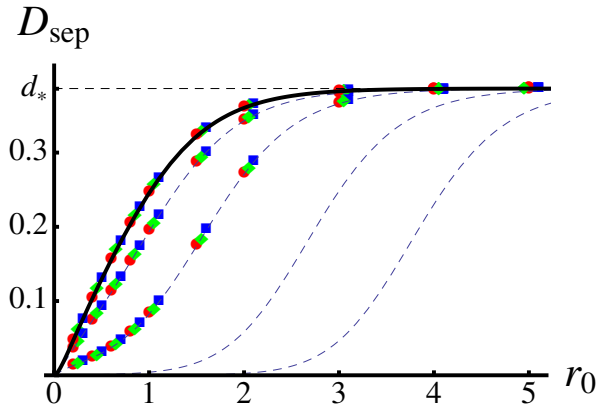

FIG. 2: Discord at separability threshold as a function of the initial squeezing. Solid black line represents the universal function $f(r_0)$, whereas the dots are the numerical solutions of the full non-Markovian dynamics in Eq. (1) for $n_T = 10$ and for three different environment spectra corresponding to ohmic, superohmic, and white noise spectral density respectively (red circles, green diamonds, and blue squares). The dashed line is the high-temperature high-squeezing limiting value $d_* \simeq 0.3863$. The gray lines denote the Markovian curves $D(\frac{1}{2} + c(t_{sep}), c(t_{sep}))$ for $n_T = 0.5, 0.1, 10^{-2}, 10^{-3}$ respectively. We also report the numerical solutions of the full non-Markovian dynamics for $n_T = 0.5, 0.1$ and the same three spectra, whereas for $n_T = 10^{-2}, 10^{-3}$ the separability threshold is definitely in the Markovian regime.

The dashed gray lines in Fig. 2 denote the Gaussian discord at separability $D_{sep} = D(\frac{1}{2} + c(t_{sep}), c(t_{sep}))$ for various (low) temperatures, whereas the dots are obtained using the full non-Markovian solutions for different environment spectra. At low temperatures the damping $\gamma(t)$ and the heating function $\Delta(t)$ become of the same order of magnitude and thus the separability threshold t_{sep} does depend on the environment spectrum. On the other hand, separability is always achieved in the Markovian regime, and thus D_{sep} is a universal quantity. For times $t \lesssim \tau_M$, there is a competition between $\gamma(t)$ and $\Delta(t)$ and in principle, one would not expect a

universal behavior. However, low temperature and weak coupling make the effect of damping and heating very weak, with appreciable perturbation of the initial state only after a long time. In other words, any dynamical effect of the interaction is taking place in the Markovian regime, thus being universal and independent on the environment spectrum. This also means that the dynamical paths in the left panel of Fig. 1 legitimately describe non-Markovian dynamical trajectories at low temperatures. More generally, any path-dependent property can be checked analytically using the set of Markovian equations and then extended to the non-Markovian regime, where an analytic approach would be unfeasible. In particular, let us introduce the rescaled time $\tau = \Gamma t$, and recall that in the Markovian regime we have $\partial_\tau \lambda = e^{-\tau}(\lambda_T - \lambda_0)$ and $\partial_\tau (\lambda\mu)^{-1} = e^{-\tau}\left[(\lambda_0\mu_0)^{-1} + 4\lambda_T\right]$, where pedices $0/T$ refer to initial/stationary state. Then, any constant of motion, e.g. $C = \lambda + y/(4\lambda\mu)$, with $y = (\lambda_T - \lambda_0)/\lambda_T + (4\mu_0\lambda_0)^{-1}$ built using the Markovian dynamical equation is a constant of motion also in the non-Markovian regime, independently on the environment spectrum. The temperature dependence disappears in the high temperatures limit.

Discussion and conclusions— We have introduced an overcomplete parameter space for two-mode symmetric Gaussian states suitable to address the decoherence effects of both Markovian and non-Markovian Gaussian maps, and to reveal universality of the dynamical paths. We have shown that Markovian paths coincide with non-Markovian ones and thus all path-dependent properties are not depending on the environment spectrum, e.g. it is possible to build constant of motions valid for both Markovian and non-Markovian maps. Dynamical paths do depend only on the initial state and on the effective temperature of the environment, and the non Markovianity behavior is only seen he velocity of running over a given path. Our results also show that a non-Markovianity measure [7, 10]]cannot be based on any geometrical characterization of the dynamical paths.

Universality is also observed for the value of discord at the separability threshold, which moreover depends on the sole initial squeezing in the high temperature limit. We also found that the geometrical constraints provided by the structure of the parameter space implies the existence of excluded regions, i.e. sets of Gaussian states which cannot be linked by any Gaussian dynamical map. Finally, we emphasize that universality of dynamical paths is related only to the assumptions of weak coupling and linear interaction and therefore it may be conjectured that it represents a more general feature, characterizing any open quantum system admitting a Markovian limit.

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